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Interior Schwarzschild Problem and Its Integration

Hanno Essén[1]

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The interior Schwarzschild metric for a static, spherically symmetric perfect fluid can be parametrized with two independent functions of the radial coordinate. These functions are easily expressed in terms of (radial) integrals involving the fluid energy density and pressure. The pressure is, however, not independent, but is determined in terms of the density by one of Einstein's equations, the Oppenheimer–Volkov (OV) equation. An approximate integral to the OV equation is presented which is accurate for slowly varying, realistic, densities, and exact in the constant-density limit. It makes it possible to find completely integrated accurate solutions to the interior Schwarzschild metric in terms of the density only. Some post-Newtonian consequences of the solution are given as well as the resulting general relativistic pressure for an energy density $\propto r^{-1/2}$.

1. INTRODUCTION

General relativity as a theory for phenomena in a curved space-time has been quite accurately tested by now. Our knowledge of the energy-momentum tensor of matter and of how matter produces gravity is, however, more limited and can only be indirectly tested. Interior solutions, i.e., solutions inside matter distributions, can be used as tools to study these problems.

Already Schwarzschild, in 1916, found the exact interior solution to Einstein's equations for the case of a constant density, and later interior solutions have been much studied because of their relevance to stellar astrophysics. Constant density is not a realistic approximation for most cases even if real densities normally vary slowly. In this article a completely integrated approximate interior solution to the Einstein equations for a static, spherically symmetric perfect fluid is given. The approximation is accurate for slowly varying density and becomes exact for a constant density (more trivially it is also exact in the Newtonian limit).

[1] Department of Mechanics, KTH, S-100 44 Stockholm, Sweden; e-mail: hanno@mech.kth.se.

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Most presentations of the interior solution give an integral solution to one of the metric components, but the other independent component is only determined via the solution of the (Tolman–) Oppenheimer–Volkov (1939) differential equation for the pressure. The main feat of this presentation is the approximate but accurate analytic solution of this equation for slowly varying densities. The second metric component is then also expressible entirely in terms of integrals involving the density.

The interior Schwarzschild solution is usually not treated in elementary texts [Schutz (1990) is an exception; even Landau and Lifshitz (1975) skim over it, but there are extensive treatments in the monographs of Weinberg (1972), Misner *et al.* (1973), and Wald (1984)]. A very compact treatment is given in Appendix B of Hawking and Ellis (1973). Sign conventions here will follow, e.g., Misner *et al.* (1973).

We start from Einstein's equations in the form

$$R_{ik} = \frac{8\pi\kappa}{c^2} J_{ik} \tag{1}$$

where $\kappa \equiv G/c^2$ and

$$J_{ik} = T_{ik} - \frac{1}{2} g_{ik} T \tag{2}$$

is the source current tensor (Essén, 1987). The perfect-fluid energy-momentum tensor is

$$T_{ik} = (\rho c^2 + p)\, u_i u_k + p g_{ik} \tag{3}$$

and its trace is

$$T = T^i_i = -\rho c^2 + 3p \tag{4}$$

In the next section, for completeness, we derive the explicit equations for the interior solution using an optimal parameterization of the metric and optimal linear combinations of the components of Einstein's tensor equation (1).

We then note that one of the equations so obtained is in fact the Oppenheimer–Volkov equation and proceed to integrate it under the assumption of slowly varying density. The result is used to derive the post-Newtonian correction to the pressure. We then discuss the nature of the interior solution and its Newtonian and post-Newtonian limits. Finally, we use the approximate integral to the Oppenheimer–Volkov (OV) equation to find an analytic estimate of the general relativisitic pressure for a density $\rho \propto r^{-1/2}$. In an Appendix a compact integration algorithm for the OV equation, assuming a given density, is presented.

2. THE INTERIOR SCHWARZSCHILD SOLUTION

We parametrize the metric so that the line element is given by

$$ds^2 = -e^{-2\kappa\delta(r)}\left(1 - 2\kappa\,\frac{m(r)}{r}\right)c^2\,dt^2 + \left(1 - 2\kappa\,\frac{m(r)}{r}\right)^{-1} dr^2 + r^2\,d\Omega^2 \tag{5}$$

Here $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$ as usual. We thus have

$$\begin{aligned} g_{tt} &= 1/g^{tt} = -e^{-2\kappa\delta(r)}\left(1 - 2\kappa\,\frac{m(r)}{r}\right) \\ g_{rr} &= 1/g^{rr} = \left(1 - 2\kappa\,\frac{m(r)}{r}\right)^{-1} \\ g_{\theta\theta} &= 1/g^{\theta\theta} = r^2, \qquad g_{\varphi\varphi} = 1/g^{\varphi\varphi} = r^2\sin^2\theta \end{aligned} \tag{6}$$

for the metric components, the off-diagonal all being zero. This parametrization has been used before (Visser, 1992, 1996) and seems to be the most suitable for the problem.

The perfect-fluid energy-momentum tensor is now taken to have components.

$$T_{tt} = g_{tt}\rho c^2, \qquad T_{rr} = g_{rr}p, \qquad T_{\theta\theta} = g_{\theta\theta}p, \qquad T_{\varphi\varphi} = g_{\varphi\varphi}p \tag{7}$$

(see e.g., Schutz, 1990). This means that the source current tensor has components

$$J_{tt} = -\frac{1}{2}\,g_{tt}\,(\rho c^2 + 3p), \qquad J_{rr} = \frac{1}{2}g_{rr}\,(\rho c^2 - p), \qquad \text{etc.} \tag{8}$$

We now introduce the notion

$$R_{(t)} = g^{tt}\,R_{tt}, \qquad R_{(r)} = g^{rr}\,R_{rr}, \qquad R_{\Omega} = \mathrm{g}^{\theta\theta}\,R_{\theta\theta} + g^{\varphi\varphi}\,R_{\varphi\varphi} \tag{9}$$

for the products of the contravariant metric components with the corresponding covariant Ricci components. We also put

$$R_{(s)} = R_{(r)} + R_{\Omega} \tag{10}$$

Using this notation, we can now express the curvatuve scalar in the alternative ways

$$R = g^{ik}\,R_{ik} = R_{(t)} + R_{(r)} + R_{\Omega} = R_{(t)} + R_{(s)} \tag{11}$$

In the same way we form

$$J_{(t)} = g^{tt} J_{tt} = -\frac{1}{2}(\rho c^2 + 3p) \tag{12}$$

$$J_{(r)} = g^{rr} J_{rr} = \frac{1}{2}(\rho c^2 - p) \tag{13}$$

$$J_\Omega = g^{\theta\theta} J_{\theta\theta} + g^{\varphi\varphi} J_{\varphi\varphi} = (\rho c^2 - p) \tag{14}$$

$$J_{(s)} = J_{(r)} + J_\Omega = \frac{3}{2}(\rho c^2 - p) \tag{15}$$

Then $J = J_{(t)} + J_{(s)} = -T = \rho c^2 - 3p$.

Since the metric, as well as the Ricci tensor and the source current tensor, are diagonal, there are four different Einstein equations (1). The two corresponding to the angular variables, however, differ only in a common factor. This leaves us with three independent equations. The three equations obtained directly from (1) are, however, not immediately transparent. It turns out that suitable linear combinations of them simplify the problem dramatically.

One clue to a suitable linear combination is suggested by intuition. Since the curvature scalar $R = R_{(t)} + R_{(s)}$ is the Lagrangian density for the gravitational field in the Einstein–Hilbert variational approach, experience from analytical mechanics suggests that changing the sign of one of the two (time and space) contributions to R should produce a Hamiltonian density. We also see that, indeed $J_{(s)} - J_{(t)} = 2\rho c^2$, so it seems promising. Computer algebra experiments aided the search for the second suitable combination.

Optimal linear combinations of the tensor component Einstein equations turn out to be

$$R_{(s)} - R_{(t)} = 16\pi\kappa\rho \tag{16}$$

$$R_{(r)} - R_{(t)} = 8\pi\kappa(\rho + p/c^2) \tag{17}$$

$$R_{(r)} + R_{(t)} = 16\pi\kappa p/c^2 \tag{18}$$

If we now denote differentiaon with respect to r by a prime ($df/dr = f'$), the above equations take the explicit form

$$m' = 4\pi r^2 \rho \tag{19}$$

$$\delta' = -4\pi r\left(\rho + \frac{p}{c^2}\right)\left(1 - 2\kappa\frac{m}{r}\right)^{-1} \tag{20}$$

$$[\delta'' - \kappa(\delta')^2]\left(1 - 2\kappa\frac{m}{r}\right) + \frac{\delta'}{r}\left(1 + \kappa\frac{m}{r} - 3\kappa m'\right) + \frac{m''}{r} = -8\pi\frac{p}{c^2} \tag{21}$$

We will refer to these as the m-equation, the δ-equation, and the p-equation, respectively. Assuming $\rho(r)$ known, we can immediately integrate the m-equation and get

$$m(r) = \int_0^r \rho(s) 4\pi s^2\, ds \tag{22}$$

Also assuming $p(r)$ known, from an equation of state $p = p(\rho)$, we can then also integrate the δ-equation directly. In the vacuum outside the body ($r > R$) where $\rho = p = 0$ one sees that $\delta = 0$ is a consistent solution. We take the this as boundary condition on δ and get

$$\delta(r) = \int_r^R \rho(s) \left(1 + \frac{p(s)}{\rho(s)c^2}\right) \left(1 - \frac{2}{c^2} \frac{Gm(s)}{s}\right)^{-1} 4\pi s\, ds \tag{23}$$

We have now found both $m(r)$ and $\delta(r)$ and thus determined the metric, provided $\rho(r)$ and $p(r)$ are known. p can, however, be written in terms of ρ, as we show below.

3. THE OPPENHEIMER–VOLKOV EQUATION

What then is the role of the p-equation (21)? Insertion of the m and δ-equations and the solution for m into this equation leads after straighrforward calculation to

$$p' = -G \frac{(\rho + p/c^2)(m + 4\pi r^3 p/c^2)}{r(r - 2\kappa m)} \tag{24}$$

which is a differential equation for p. This is the so-called Oppenheimer–Volkov (OV) equation, the general relativistic equation for hydrostatic equilibrium (Oppenheimer and Volkov, 1939). It is normally obtained from the equation of motion $T^{ik}_{;k} = 0$. Here it seen to follow from the Einstein equations directly.

One notes that, compared to the Newtonian hydrostatic equilibrium

$$p_0' = -G\frac{\rho m}{r^2} \tag{25}$$

the general relativistic corrections all steepen the pressure gradient. To see more clearly how the two differ qualitatively, one can rearrange (24) a bit. If we introduce the mean density $\bar{\rho}$ inside radius r,

$$\bar{\rho}(r) = m(r)/\,(4\pi r^3/3) \tag{26}$$

we find the form

$$p' = p_0' \left(1 + \frac{p}{\rho c^2}\right)\left(1 + 3\frac{p}{\bar{\rho} c^2}\right)\left(1 + 2\frac{p_0' r}{\rho c^2}\right)^{-1} \tag{27}$$

This form shows that all corrections are essentially pressure divided by energy density.

The above form for the OV equation is easily rewritten

$$\frac{dp}{[1 + p/(\rho c^2)][1 + 3p/(\bar{\rho} c^2)]} = p_1'(r)\, dr \tag{28}$$

where we define

$$p_1'(r) = p_0' \left(1 + 2\frac{p_0' r}{\rho c^2}\right)^{-1} = \frac{-G\rho(r)m(r)}{r^2 - 2\kappa r m(r)} \tag{29}$$

(28) immediately shows that if the density is constant (and therefore the average density, too) we have separated the differential equation and can integrate it directly. This fact lies behind Schwarzschild's exact constant-density solution.

Should the density not be constant, one can, of course, proceed and integrate anyway, as if it were. The error involved must arise from the r-derivative of the left-hand side. We find (here we must consider p to be an independed variable) that

$$\frac{d}{dr}\frac{1}{(1 + p/\rho c^2)\,(1 + 3p/\bar{\rho} c^2)} = \frac{(1 + 3p/\bar{\rho} c^2)\,(p/\rho c^2)\, d\ln\rho/dr + (1 + p/\rho c^2)\,(3p/\bar{\rho} c2)\, d\ln\bar{\rho}/dr}{(1 + p/\rho c^2)^2\,(1 + 3p/\bar{\rho} c^2)^2} \tag{30}$$

This derivative is thus essentially proportional to the product of the relativistically small $p/(\rho c^2)$ and the derivative of the logarithm of ρ. If ρ is slowly varying, the derivative of the logarithm naturally is very small. In conclusion the result of direct integration of equation (28) can be expected to be excellent under most realistic circumstances. It is analogous to phase-integral approximations common in quantum mechanics.

Integrating (28), we now get

$$\int_0^{p(r)} \frac{dq}{[1 + q/(\rho c^2)][1 + 3q/(\bar{\rho} c^2)]} = \int_R^r p_1'(s)\, ds \tag{31}$$

The left-hand q-integral can be done, so, defining $\mathcal{P}_1(r)$ to be the primitive function of $p_1'(r)$, we find

$$\frac{\ln[1 + p(r)/(\rho c^2)] - \ln[1 + 3p(r)/(\bar{\rho} c^2)]}{1/(\rho c^2) - 3/(\bar{\rho} c^2)} = \mathcal{P}_1(r) - \mathcal{P}_1(R) \qquad (32)$$

This can now be solved for $p(r)$. If we put $p_1(r) = \mathcal{P}_1(r) - \mathcal{P}_1(R)$, we obtain $(0 \leq R)$

$$p(r) = \rho c^2 \frac{\exp[(1 - 3\rho/\bar{\rho})\, p_1/(\rho c^2)] - 1}{1 - (3\rho/\bar{\rho}) \exp[(1 - 3\rho/\bar{\rho}) p_1/(\rho c^2)]} \qquad (33)$$

where

$$p_1(r) = \int_r^R \frac{G\rho(s) m(s)\, ds}{s^2 - 2\kappa s m(s)} \qquad (34)$$

Since $\bar{\rho}(r)$ and $m(r)$ both are given by $\rho(r)$, we see that this equation gives a general relativistic $p(r)$ for an arbitrarily given, not too rapidly varying, density ρ. Should the present approximation not be sufficient, a simple algorithm for the full solution is given in the Appendix.

To first order, the solution (33) of the OV equation is

$$p(r) \approx p_1(r) \left[1 + \frac{1}{2c^2} \left(\frac{1}{\rho} + \frac{3}{\bar{\rho}} \right) p_1(r) + \ldots \right] \qquad (35)$$

Equation (34) gives, to first order,

$$p_1(r) \approx p_0(r) + \int_r^R \frac{2G^2}{c^2} \frac{\rho m^2}{s^3}\, ds + \ldots \qquad (36)$$

where $p_0(r)$ is the Newtonian pressure. Inserting this into (35), we get

$$p(r) \approx p_0(r) + \frac{1}{2c^2} \left(\frac{1}{\rho} + \frac{3}{\bar{\rho}} \right) p_0^2(r) + \int_r^R \frac{2G^2}{c^2} \frac{\rho m^2}{s^3}\, ds + \ldots \qquad (37)$$

and so find the first-order general relativistic corrections to the Newtonian pressure.

4. THE NEWTONIAN AND POST-NEWTONIAN LIMITS OF THE INTERIOR SOLUTION

With the accurate result (33) for the general relativistic pressure $p(r)$ inserted into the expression (23) for the function $\delta(r)$ we have finally a completely integrated solution to the interior Schwarzschild metric in terms of an (an arbitrary) $\rho(r)$ only. Using these formulas; one can generate accurate general relativistic solutions (other than the constant-density one). One will be presented in the next section.

Let us first, however, have a look at the meaning of the function $\delta(r)$ in the metric. In order to do this, it is useful to note that one can find the Newtonian potential ϕ from the metric through the identifiction

$$g_{tt}(r) = -(1 + \frac{2}{c^2}\phi(r)) \tag{38}$$

Outside ($r > R$) the body this is exact, both as a Newtonian and as a Schwarzschild result, since there one has $\phi(r) = -Gm(R)/r$ in both cases. Inside the body ($0 \leq r < R$) things are more complex. In the Newtonian theory the potential at r can be thought of as having two parts inside a spherical body. One part, $\phi_<$, from the matter inside the radius r, is given by

$$\phi_<(r) = -G\frac{m(r)}{r} = -G\frac{1}{r}\int_0^r \rho(s)4\pi s^2\, ds \tag{39}$$

The other part, $\phi_>$, comes from the matter outside r and is given by

$$\phi_>(r) = -G\,\delta_0(r) = -G\int_r^R \rho(s)4\pi s\, ds \tag{40}$$

and the full Newtonian potential is $\phi_0 = \phi_< + \phi_>$.

If we now consider equation (23) for the function $\delta(r)$, we see that in the Newtonian limit it gives δ_0 as given by (40). In this limit we can also approximate the exponential in the metric component g_{tt} of equation (6), and thus find that, in the Newtonian limit, the Schwarzschild solution can be written

$$g_{tt}(r) = -(1 + \frac{2}{c^2}\,\phi_0(r)) \tag{41}$$

$$g_{rr}(r) = (1 + \frac{2}{c^2}\,\phi_<(r))^{-1} \tag{42}$$

Outside the body $\phi_< = \phi_0 = \phi$, so the difference vanishes and both expressions are exact; inside, only g_{rr} is exact. In spite of the many treatments of the interior Schwarzschild solution in the literature, this simple fact is rarely, if ever, mentioned.

The Newtonian acceleration $g_0(r)$ at r is given by

$$g_0(r) = \phi_0'(r) = G\frac{m(r)}{r^2} \tag{43}$$

and depends only on the mass inside r. The role of the $\phi_>$ part of the potential is thus essentially to cancel the $-Gm'/r$ contribution coming from $\phi'_<$. It is interesting to note that in formula (41) all approximation comes from the $\phi_>$ part of ϕ, i.e., it should really be in an exponential and should really have general relativistic contributions from pressure and nonconstant metric. Deeper reasons why the exterior solution does not depend on the pressure have been given by Deser and Laurent (1968).

Comparing (38) and (6), we can, using our expressions for m and δ found above, calculate the first-order general relativistic correction to ϕ. This correction has calculated before (Chandrasekhar 1965), but usually by more complicated methods. One finds

$$\phi \approx \phi_0 + \phi_1$$
$$= \phi_0 - \left(\frac{G}{c}\right)^2 \left[\int_r^R \left(\frac{p_0}{G} + 2\,\frac{\rho m}{s} \right) 4\pi s\, ds - 2\left(\frac{\delta_0 m}{\mathrm{r}} + \frac{1}{2}\,\delta_0^2 \right) \right] \tag{44}$$

where $\phi_0 = -G[(m/r) + \delta_0]$. The correction ϕ_1 is seen to become zero at the surface $r = R$. A simple calculation shows that this also is true for the first-order correction to the acceleration $g_1(R) = (d\phi_1/dr)_{r=R} = 0$. The gravity gradient, i.e., the derivative of g_1 is, however, nonzero. One finds

$$\left(\frac{dg_1}{dr}\right)_{r=R} = \left(\frac{d^2\phi_1}{dr^2}\right)_{r=R} = \left(\frac{G}{c}\right)^2 \frac{4\pi\rho(R)}{R}\, m(R) \tag{45}$$

Since $g_0'(R) = -(2g_0/R) + 4\pi G\rho$, one thus finds that the gravity gradient, with first-order general relativistic correction is

$$\left(\frac{dg}{dr}\right)_{r=R} \approx -\frac{2g(R)}{R} + 4\pi G\left(1 + \frac{G}{c^2}\frac{m(R)}{R}\right)\rho(R) \tag{46}$$

at the surface of a body. For the earth $GM/(c^2R) \approx 10^{-9}$, so this change in apparent G value is presently completely beyond experimental detection (in good agreement with conventional wisdom).

5. AN APPROXIMATE ANALYTIC INTERIOR GENERAL RELATIVISTIC PRESSURE

An approximate but accurate solution for the pressure and the metric, given a slowly varying ρ, is, according to the results above, calculated as follows.

1. Given ρ, calculate m according to equation (22). This also gives the average density $\bar{\rho}$ according to (26).

2. Using ρ and m, calculate p_1, of equation (34).
3. Using p_1, ρ, and $\bar{\rho}$, find the general relativistic pressure p according to (33).
4. Finally, find the function δ using ρ, p, and m in equation (23).

With this scheme in mind one might try to find a density that gives analytic, closed-form, expressions for as many interesting quantities as possible. The case of a constant density was done already by Schwarzschild. A more interesting exact interior solution has been found by Buchdahl (1981). Below I will present a solution that is not exact, but which gives an accurate analytic result for the general relativisitic pressure p (except near the origin).

Simple considerations lead one to suspect that a simple Newtonian pressure p_0 is a good starting point. We thus try the density

$$\rho(r) = \begin{cases} ar^{-1/2} & \text{for} \quad 0 \le r \le R \\ 0 & \text{for} \quad R < r \end{cases} \tag{47}$$

If we introduce $\eta = G8\pi a/(c^2 5)$ and the dimensionless variable x through $x = \eta^{2/3} r$, this gives

$$\rho(r) = a\eta^{1/3} x^{-1/2} \tag{48}$$

$$m(r) = \kappa^{-1}\eta^{-2/3}x^{5/2} \tag{49}$$

$$\bar{\rho}(r) = (6/5)\rho = (6/5)a\eta^{1/3}\, x^{-1/2} \tag{50}$$

$$p_0'(r) = -c^2 a\eta \tag{51}$$

$$p_0(r) = c^2 a\eta^{1/3}\,(X - x) \tag{52}$$

where $X = \eta^{2/3}\, R$ represents the surface. We now introduce the function

$$f(x) = \int_0^x \frac{ds}{1 - s^{3/2}} \tag{53}$$

since it can be used to express p_1 as follows:

$$p_1(r) = c^2 a\eta^{1/3}[f(X) - f(x)] \tag{54}$$

Series expansion and term by term integration of f gives the fairly simple result

$$f(x) = 2x \sum_{k=0}^{\infty} \frac{(\sqrt{x})^{3k}}{2 + 3k} \tag{55}$$

Maple (see, e.g., Nicolaides and Walkington, 1996) gives the somewhat longer analytic expression

$$
\begin{aligned}
f(x) = \frac{1}{3}\Bigg\{ & 2\,\mathrm{arctanh}(\sqrt{x}) - \ln(1-x) - \frac{\pi\sqrt{3}}{6} \\
& + \frac{1}{2}\left[\ln(1+x+x^2) + \ln\left(\frac{1+\sqrt{x}+x}{1-\sqrt{x}+x}\right)\right] \\
& + \sqrt{3}\left[\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) - \arctan\left(\frac{2}{\sqrt{3}}\sqrt{x} - \frac{1}{\sqrt{3}}\right)\right. \\
& \left. - \arctan\left(\frac{2}{\sqrt{3}}\sqrt{x} + \frac{1}{\sqrt{3}}\right)\right]\Bigg\}
\end{aligned} \tag{56}
$$

For small (nonrelativisitic) x values the series should suffice, but since we are interested in general relativistic effects here we need good results for x near unity. The analytic expression should then be useful.

We now put our results into formula (33) and get the general relativistic pressure in the form

$$p(r) = c^2\, a\eta^{-1/3}\Pi_X(x) \tag{57}$$

where

$$\Pi_X(x) = \frac{1}{\sqrt{x}}\,\frac{\exp\{(3/2\sqrt{x}[f(x) - f(X)]\} - 1}{1 - (5/2)\exp\{(3/2)\sqrt{x}[f(x) - f(X)]\}} \tag{58}$$

One notes that

$$\lim_{x\to 0} \Pi_x(x) = f(X) \tag{59}$$

but near the origin $x^{-1/2}$ is not slowly varying and thus the solution cannot be trusted as an accurate general relativistic solution in that limit.

The function $\Pi_X(x)$ is plotted for four different values of X in Figs. 1–4 (the normalized variable x/X is on the horizontal axis). The lowest curve in each diagram is the Newtonian pressure ($p_0 \propto X - x$). The middle curve is the pressure p_1 of equation (54). One notes that the pressure p develops an unphyiscal hump and thus has positive derivative near the origin. This is due to the fact that the density is not slowly varying near the origin. At $X \approx 0.8388$ a singularity develops in p.

6. CONCLUSIONS

The solution to the Schwarzschild interior metric given in this paper differs from those found in the literature only in minor variations. It seems, however, pedagogically appealing and the natural connection with the New-

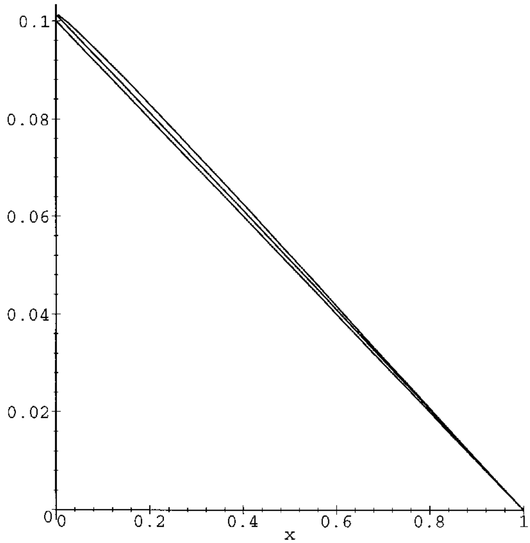

Fig. 1. The Newtonian pressure $p_0 \propto X - x$ (bottom curve), the pressure $p_1 \propto f(X) - f(x)$ (middle curve), and the approximate general relativistic pressure $p \propto \prod_X(x)$ (top curve). The value of X is 0.1 The horizontal axis is the normalized x/X.

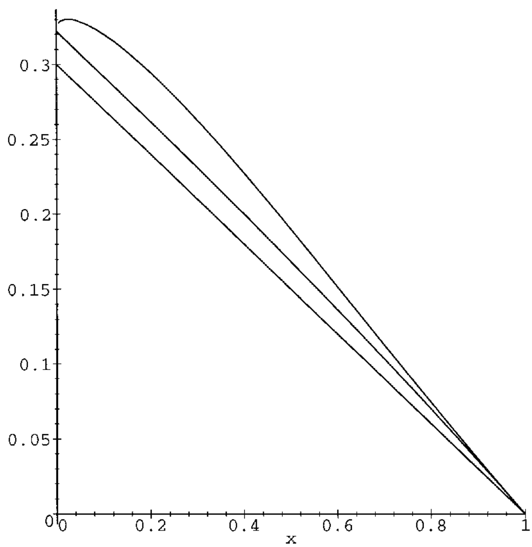

Fig. 2. Same as Fig. 1, but for $X = 0.3$.

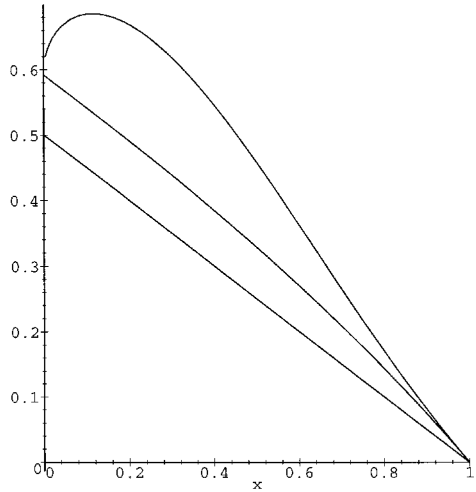

Fig. 3. Same as Fig. 1, but for $X = 0.5$. The unphysical hump in the pressure p is due to the rapid variation of the density near the origin.

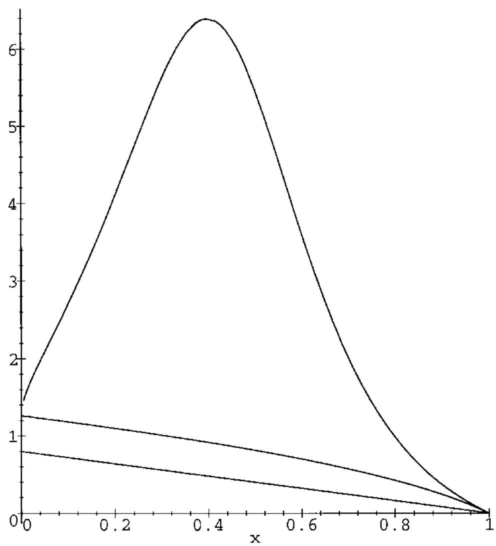

Fig. 4. Same as Fig. 1, but for $X = 0.8$. The pressure p is becoming very unphysical in the interior. At $X \approx 0.8388$ a singularity develops in p.

tonian result for the interior that I have stressed above seems to have some novelty. It is perhaps noteworthy that the OV equation is simply one of Einstein's equations in my formalism. The divergencelessness of the Einstein tensor which gives $T^{ik}_{;k} = 0$, which is normally used to get the OV equation, requires after all an extra covariant differentiation.

The approximate solution of the OV equation that is exact in the constant-density as well as the nonrelativistic limit should be useful for qualitative studies. The explicit inverse square root density investigated in the last section does not really do justice to the approximate solution to the OV equation since this density definitely violates the assumption of slowly varying density near the origin. It was, however, the only reasonably physical density that readily gave analytic closed-form expressions. The singularity in the pressure when the Schwarzschild radius is approached is, however, a feature that it has in common with the constant-density solution and exact solutions, even if it does not come at the origin in my approximation.

APPENDIX. ALGORITHM INTEGRATING THE OV EQUATION FOR A GIVEN DENSITY

The integration of the OV equation (24) is usually treated in the literature in connection with stellar modeling. One then assumes an equation of state $p = p(\rho)$ and integrates three coupled equations to get both p and ρ. This is clearly described in §23.7 of Misner *et al.* (1973).

If one is interested only in finding the pressure given a density, a very elegant algorithm can be found. Equation (27); using (29), gives

$$p' = p_1' \left(1 + \frac{p}{\rho c^2}\right)\left(1 + \frac{3p}{\rho c^2}\right) \tag{A1}$$

We do not know p, so we cannot integrate, but we can get p_1 [see equation (34)] and find an approximation to p by integrating

$$p_2' = p_1' \left(1 + \frac{p_1}{\rho c^2}\right)\left(1 + \frac{3p_1}{\rho c^2}\right) \tag{A2}$$

to

$$p_2(r) = \int_r^R p_1' \left(1 + \frac{p_1}{\rho c^2}\right)\left(1 + \frac{3p_1}{\rho c^2}\right) ds \tag{A3}$$

This leads to the iteration formula

$$p_{k+1}(r) = \int_r^R p_1' \left(1 + \frac{p_k}{\rho c^2}\right)\left(1 + \frac{3p_k}{\bar{\rho} c^2}\right) ds \tag{A4}$$

where p_1, defined in terms of p_0 in equation (34), is the starting approximation. Assuming convergence, the procedure leads to a self-consistent general relativistic pressure $p(r) = \lim_{k\to\infty} p_k(r)$.

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